WILLIAM D. LAW, JR. LEARNING COMMONS

Logarithms

A **logarithm** of a given number x, is the exponent required for the base a, to be raised to in order to produce that number x.

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

Note that $\Leftrightarrow$ means "$is\ equivalent\ to$"

Logarithmic and Exponential Form

Change logarithm equations to exponential form or exponential equations to logarithmic form using the definition of a logarithm.

Example: Given $4^{3/2} = 8$, change the equation to logarithmic form.

Solution:

Compare the equation to the definition and rewrite it.

Definition: $\log_a x = y \Leftrightarrow a^y = x$ ⟷ Notice that $a = 4, x = 8$, and

Given: $4^{3/2} = 8$ $y = \frac{3}{2}$, respectively.

Therefore, using the definition: $4^{3/2} = 8 \Leftrightarrow \mathbf{\log_4 8 = \frac{3}{2}}$

Example: Given $\log_{25} 5 = \frac{1}{2}$, change the equation to exponential form.

Solution:

Compare the equation to the definition and rewrite it.

Definition: $\log_a x = y \Leftrightarrow a^y = x$ ⟷ Notice that $a = 25, x = 5$, and

Given: $\log_{25} 5 = \frac{1}{2}$ $y = \frac{1}{2}$, respectively.

Therefore, using the definition: $\log_{25} 5 = \frac{1}{2} \Leftrightarrow \mathbf{25^{1/2} = 5}$

Solving Logarithm and Exponential Equations

Evaluate logarithmic equations by using the definition of a logarithm to change the equation into a form that can then be solved.

Example: Given $3^{x-1} = 7$, solve for x.

Solution:

Step 1: Set up the equation and use the definition to change it.

Definition: $\log_a x = y \iff a^y = x$
Given $3^{x-1} = 7$

Notice 3 is the base or a,
and 7 is the given number.

$3^{x-1} = 7 \iff \log_3 7 = x - 1$

Step 2: Now use the properties of logarithms to solve.

Recall the Change of Base Property:

$$\log_a b = \frac{\log b}{\log a}$$

Apply it to $\log_3 7$.

$$\log_3 7 = \frac{\log 7}{\log 3}$$

Step 3: Use the order of operations to finish solving for x.

$$x - 1 = \frac{\log 7}{\log 3}$$

$$\boldsymbol{x = \frac{\log 7}{\log 3} + 1}$$

Example: Given $\log_6(x + 2) = 3$, solve for x.

Solution:

Step 1: Set up the equation and use the definition to change it.

Definition: $\log_a x = y \iff a^y = x$
Given $\log_6(x + 2) = 3$
Notice 6 is the base or a,
and 3 is the exponent or y.

$\log_6(x + 2) = 3 \iff 6^3 = x + 2$

Step 2: Now use the order of operations to solve.

$6^3 = x + 2$
$216 = x + 2$
$214 = x$

$\boldsymbol{x = 214}$

Expanding and Simplifying Logarithms

To expand or simplify logarithms, utilize the various properties of logarithms in conjunction with the definition.

Example: Given $\log_3\left(\frac{9x^2}{\sqrt{x^2+1}}\right)$, expand the logarithm.

Solution:

Step 1: Expand the expression using the properties of logarithms.

Recall the Logarithm Multiplication and Division Properties:
$\log_a mn = \log_a m + \log_a n$
$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$
Apply them to $9x^2$ and $\sqrt{x^2+1}$.

Given $\log_3\left(\frac{9x^2}{\sqrt{x^2+1}}\right)$:

$\Rightarrow \log_3 9 + \log_3 x^2 - \log_3\left(\sqrt{x^2+1}\right)$

Step 2: Now simplify further using the properties of logarithms and the definition.

Recall the Logarithm for Powers Property:
$\log_a x^c = c\log_a x$
Apply it to the x^2 and $\log_3\left(\sqrt{x^2+1}\right)$.

$\log_3 9 + \log_3 x^2 - \log_3\left(\sqrt{x^2+1}\right)$
$\Rightarrow \log_3 9 + \log_3 x^2 - \log_3(x^2+1)^{1/2}$
$\Rightarrow \log_3 9 + 2\log_3 x - \frac{1}{2}\log_3(x^2+1)$

By definition, $\log_3 9 = 2$ since $3^2 = 9$, so our final answer becomes:
$\mathbf{2 + 2\log_3 x - \frac{1}{2}\log_3(x^2+1)}$

Example: Write $3\log_2 y - \log_2 x - 7\log_2 z$ as a single logarithm.

Solution:

To simplify the expression, work backwards with the logarithmic properties.

Step 1: Use the Logarithm for Powers Property where appropriate.

Given: $3\log_2 y - \log_2 x - 7\log_2 z$
Notice that it can be applied to $3\log_2 y$ and $7\log_2 z$.

$3\log_2 y - \log_2 x - 7\log_2 z$
$\Rightarrow \log_2 y^3 - \log_2 x - \log_2 z^7$

Step 2: Simplify using the Logarithm Multiplication and Division Properties. Use the order of operations as a guide.

$\log_2 y^3 - \log_2 x - \log_2 z^7$
$\Rightarrow \log_2 y^3 - (\log_2 x + \log_2 z^7)$
$\Rightarrow \log_2 y^3 - \log_2 xz^7$
$\Rightarrow \mathbf{\log_2 \frac{y^3}{xz^7}}$

This instructional aid was prepared by the Tallahassee Community College Learning Commons.

Solving Expanded Logarithms

Solving expanded logarithms requires applying the definition of logarithms and all the logarithm properties as needed.

Example: Given $\ln(x-2)+\ln(x-3)=\ln(2x+24)$, solve for x.

Solution:

Note: $\ln(x-2)$ is only valid if $x \geq 2$, $\ln(x-3)$ is only valid if $x \geq 3$, and $\ln(2x+24)$ is only valid if $x \geq -12$. For the equation to be valid, all conditions must be met, so $x \geq 3$.

Step 1: Simplify the left side of the equation using the multiplication and division properties of logarithms.

$\ln(x-2)+\ln(x-3)=\ln(2x+24)$

$\Rightarrow \ln(x-2)(x-3)=\ln(2x+24)$
$\Rightarrow \ln(x^2-5x+6)=\ln(2x+24)$

Step 2: Use logarithm properties.
Recall logarithm properties of bases:
$\ln e^x = x$ and $e^{\ln x}=x$

$\ln(x^2-5x+6)=\ln(2x+24)$
Let both sides of the equation become the exponent of the base e, and apply the property.

$\Rightarrow e^{\ln(x^2-5x+6)}=e^{\ln(2x+24)}$
$\Rightarrow x^2-5x+6=2x+24$

Step 3: Combine like terms to solve for x.
$x^2-5x+6=2x+24$
$\Rightarrow x^2-7x-18=0$
$\Rightarrow (x-9)(x+2)=0$
$x=9,-2$

Step 4: Check your answers. Recall that every logarithm must meet the conditions for the answer to be correct.

For $x=9$
$\ln((9)-2)+\ln((9)-3)=\ln(2(9)+24)$
$\Rightarrow \ln(7)+\ln(6)=\ln(42)$
$\Rightarrow \ln(7\cdot 6)=\ln(42)$ ⇢ This is valid!

For $x=-2$
Since $-2 \ngeq 3$, it does not meet all the conditions, and is not valid.

Therefore: $\boldsymbol{x=9}$

Practice Exercises:

1. Given $\log_4(-x)+\log_4(6-x)=2$, Solve for x.
2. Expand $\log_2\left(\frac{x}{\sqrt{x^2-1}}\right)$ completely.
3. Write the following as a single logarithm: $2\log_3 x+4-8\log_3 y$

Answers:

1. $x=-2$
2. $\log_2 x-\frac{1}{2}\log_2(x-1)-\frac{1}{2}\log_2(x+1)$
3. $\log_3\frac{81x^3}{y^8}$